Tightness for processes with fixed points of discontinuities and applications in varying environment

Vincent Bansaye* Thomas G. Kurtz† Florian Simatos‡

Abstract

We establish a sufficient condition for the tightness of a sequence of stochastic processes. Our condition makes it possible to study processes with accumulations of fixed times of discontinuity. Our motivation comes from the study of processes in varying or random environment. We demonstrate the usefulness of our condition on two examples: Galton Watson branching processes in varying environment and logistic branching processes with catastrophes.

Keywords: tightness; fixed points of discontinuity; varying environment.
AMS MSC 2010: 60B10.
Submitted to ECP on March 1, 2016, final version accepted on June 13, 2016.
Supersedes arXiv:1409.5215.

1 Main result: statement and discussion

Statement

Let $(\mathfrak{X}, d)$ be a separable, complete metric space and $D_{\mathfrak{X}}$ be the space of càdlàg functions $f : [0, \infty) \to \mathfrak{X}$. The space $D_{\mathfrak{X}}$ is endowed with the Skorohod J_1 topology, and we write $f_n \to f$ for convergence in this space and $X_n \Rightarrow X$ for the corresponding weak convergence of stochastic processes. See, for instance, Billingsley [3] for more details. For $f \in D_{\mathfrak{X}}$ and $t \geq 0$ we write $f(t-) = \lim_{s \uparrow t} f(s)$ (with the convention $f(t-) = f(0)$ if $t = 0$) and $\Delta f(t) = d(f(t), f(t-))$. The above definitions and notation apply to the case $\mathfrak{X} = \mathbb{R}$ and d is the Euclidean distance, in which case we denote by $\mathcal{V}$ the set of càdlàg functions $f \in D_{\mathbb{R}}$ which are non-decreasing.

For each $n \geq 1$, we consider a càdlàg process $X_n = (X_n(t), t \geq 0)$ adapted to a filtration $\{\mathcal{F}_t^n, t \geq 0\}$. Unless otherwise specified, the identities and inequalities stated for the processes hold almost surely (a.s.).

Theorem 1.1. *Assume that:*

A1) *For each $T, \varepsilon > 0$, there exists a compact set K of $\mathfrak{X}$ such that*

$$\liminf_{n \to \infty} \mathbb{P}\left(X_n(t) \in K, \forall t \leq T\right) \geq 1 - \varepsilon. \tag{1.1}$$

*CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France.
E-mail: vincent.bansaye@polytechnique.edu
†Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706-1388.
E-mail: kurtz@math.wisc.edu
‡ISAE Supaero, Département DISC, 10 avenue Edouard Belin, BP 54032, 31055 Toulouse Cedex 4, France.
E-mail: florian.simatos@isae.fr

A2) *There exist stochastic processes* $F_n, F \in \mathcal{V}$ *such that* $\sigma(F_n) \subset \mathcal{F}_0^n$ *and* $F_n \Rightarrow F$ *and* $\beta > 0$ *such that for every* $n \geq 1$ *and every* $0 \leq s \leq t$*,*

$$\mathbb{E}\left[1 \wedge d\left(X_n(t), X_n(s)\right)^\beta \mid \mathcal{F}_s^n\right] \leq F_n(t) - F_n(s). \tag{1.2}$$

Then the sequence $(X_n, n \geq 1)$ *is tight in* $D_{\mathfrak{X}}$*.*

One easily checks, for instance by going back to the Arzelà–Ascoli characterization of tightness, that in presence of the compact containment condition A1 the sequence (X_n) is tight if and only if for every compact set K, the sequence (X_n) stopped upon its first exit of K is tight. Thus we have the following simple extension of the previous theorem.

Corollary 1.2. *For* $K \subset \mathfrak{X}$ *let* $T_n^K = \inf\{t \geq 0 : X_n(t) \notin K\}$*. Assume that the compact containment condition* A1 *holds and that:*

A2') *For every compact subset* $K \subset \mathfrak{X}$*, there exist stochastic processes* $F_n, F \in \mathcal{V}$ *such that* $\sigma(F_n) \subset \mathcal{F}_0^n$ *and* $F_n \Rightarrow F$ *and* $\beta > 0$ *such that for every* $n \geq 1$ *and every* $0 \leq s \leq t$*,*

$$\mathbb{E}\left[1 \wedge d\left(X_n(t \wedge T_n^K), X_n(s \wedge T_n^K)\right)^\beta \mid \mathcal{F}_s^n\right] \leq F_n(t) - F_n(s). \tag{1.3}$$

Then the sequence $(X_n, n \geq 1)$ *is tight in* $D_{\mathfrak{X}}$*.*

We finally mention a second direct extension which is useful for the study of Galton–Watson processes in varying environments, see below.

Corollary 1.3. *Assume that the compact containment condition* A1 *holds, and that:*

A2") *There exist stochastic processes* $F_n, F \in \mathcal{V}$ *such that* $\sigma(F_n) \subset \mathcal{F}_0^n$ *and* $F_n \Rightarrow F$ *and* $\beta, \eta > 0$ *such that for every* $n \geq 1$ *and every* $0 \leq s \leq t$ *such that* $F_n(t) - F_n(s) \leq \eta$*,*

$$\mathbb{E}\left[1 \wedge d\left(X_n(t), X_n(s)\right)^\beta \mid \mathcal{F}_s^n\right] \leq F_n(t) - F_n(s). \tag{1.4}$$

Then the sequence $(X_n, n \geq 1)$ *is tight in* $D_{\mathfrak{X}}$*.*

Proof. Let $\tilde{F}_n(t) = F_n(t)/1 \wedge \eta$: then the inequality

$$\mathbb{E}\left[1 \wedge d\left(X_n(t), X_n(s)\right)^\beta \mid \mathcal{F}_s^n\right] \leq \tilde{F}_n(t) - \tilde{F}_n(s)$$

holds for every $0 \leq s \leq t$. Indeed, if $F_n(t) - F_n(s) \leq \eta$ then this follows from (1.4) by dividing by $1 \wedge \eta \leq 1$, while if $F_n(t) - F_n(s) \geq \eta$ then $\tilde{F}_n(t) - \tilde{F}_n(s) \geq 1$ and the inequality is trivially satisfied. Thus we can invoke Theorem 1.1 to conclude. □

Discussion

If F were continuous, then the result would follow immediately from Theorem $3.8.6$ of [5] by taking $\gamma_\alpha(\delta) = \sup_{0 \leq t \leq t+u \leq T, u \leq \delta}(F_\alpha(t+u) - F_\alpha(t))$ (see also Theorem 4.20 of [8]). But, of course, the point of Theorem 1 of the paper is that F is not continuous. Allowing F to be discontinuous is motivated by the study of processes in varying environment, where, typically, non-critical environments can create fixed times of discontinuity which translate to discontinuities of F. When there are only finitely many fixed times of discontinuity, one can prove tightness on time-intervals without fixed times of discontinuity and then "glue" the pieces together (using for instance Lemma 2.2 in Whitt [13]). However, this approach seems more challenging when fixed times of discontinuity can accumulate, and even be dense. The interest of Theorem 1.1 is to allow for such cases and we provide motivations and applications in Sections 3 and 4.

2 Proof of Theorem 1.1

First step

We start with some preliminary remarks and the introduction of some auxiliary functions γ_n, Y_n and G_n. First, note that we can assume without loss of generality that F_n and F satisfy the following three properties:

i) $F_n(t) - F_n(s), F(t) - F(s) \geq t - s$ for any $0 \leq s \leq t$;

ii) $F_n(0) = F(0) = 0$;

iii) F_n and F are unbounded.

Indeed, otherwise we can simply replace F_n and F by $\tilde{F}_n(t) = F_n(t) - F_n(0) + t$ and $\tilde{F}(t) = F(t) - F(0) + t$, so that $F_n(t) - F_n(s) = \tilde{F}_n(t) - \tilde{F}_n(s) - (t-s) \leq \tilde{F}_n(t) - \tilde{F}_n(s)$ and assumption A2 still holds with $\tilde{F}_n$ in place of F_n. In particular, F_n and F are strictly increasing and unbounded.

In the sequel, we therefore assume that F_n satisfies these three properties. For $f \in \mathcal{V}$ and unbounded we define $f^{-1} \in \mathcal{V}$ the function defined by $f^{-1}(t) = \inf\{s \geq 0 : f(s) > t\}$. We will consider in particular $\gamma_n = F_n^{-1}$, which satisfies the following properties (see Section 13.6 in Whitt [14]):

i) $\gamma_n(0) = 0$ and γ_n is Lipschitz continuous and unbounded;

ii) $\gamma_n^{-1} = F_n$ and $\gamma_n \circ \gamma_n^{-1} = \mathrm{Id}$, with Id the identity function $\mathrm{Id}(t) = t$;

iii) $\gamma_n(t)$ is $\mathcal{F}_0^n$-measurable and hence is a $\{\mathcal{F}_t^n\}$-stopping time.

We further define the càdlàg processes

$$Y_n(t) = \lim_{s \to t+} X_n\left(\gamma_n(s)-\right) \text{ and } G_n(t) = \lim_{s \to t+} F_n\left(\gamma_n(s)-\right). \tag{2.1}$$

Since $\gamma_n \circ F_n = \mathrm{Id}$ and $\gamma_n(F_n(a)) < \gamma_n(s)$ if $s > F_n(a)$ and X_n is right continuous,

$$X_n = Y_n \circ F_n.$$

The proof now consists in proving that Y_n is tight and to derive the tightness of X_n using Lemma 2.5 of [9].

First, the compact containment condition for (Y_n) simply follows from the identity

$$\mathbb{P}\left(Y_n(t) \in K, \forall t \leq T\right) = \mathbb{P}\left(X_n(t) \in K, \forall t \leq \gamma_n(T)\right)$$

together with the facts that the sequence $(\gamma_n(T))$ is bounded and that (X_n) satisfies by assumption the compact containment condition A1.

Second step

We now prove that the sequence (Y_n) is tight. Let in the sequel $q(x,y) = 1 \wedge d(x,y)$. Note that since $\gamma_n(t)$ is $\mathcal{F}_0^n$-measurable for any $t \geq 0$, (1.2) implies that for $0 < s < t$ and $0 < \delta < \gamma_n(s)$,

$$\mathbb{E}\left[q(X_n(\gamma_n(t)-\delta), X_n(\gamma_n(s)-\delta))^\beta | \mathcal{F}^n_{\gamma_n(s)-\delta}\right] \leq F_n(\gamma_n(t)-\delta) - F_n(\gamma_n(s)-\delta),$$

and letting $\delta \to 0$

$$\mathbb{E}\left[q(X_n(\gamma_n(t)-), X_n(\gamma_n(s)-))^\beta | \mathcal{F}^n_{\gamma_n(s)-}\right] \leq F_n(\gamma_n(t)-) - F_n(\gamma_n(s)-). \tag{2.2}$$

Again, we are using that for each t, $\gamma_n(t)$ is a predictable stopping time. Define $\mathcal{G}_s^n = \cap_{r>s}\mathcal{F}^n_{\gamma_n(r)-}$. Taking decreasing limits in (2.2), we have

$$\mathbb{E}\left[q\left(Y_n(t), Y_n(s)\right)^\beta \mid \mathcal{G}_s^n\right] \le G_n(t) - G_n(s)$$

which implies in particular that $q(Y_n(t), Y_n(s)) \le \mathbb{1}_{\{G_n(t)>G_n(s)\}}$. Thus for any $0 \le v \le t$ we have

$$\mathbb{E}\left[q(Y_n(t+u), Y_n(t))^\beta \mid \mathcal{G}_t^n\right] q(Y_n(t), Y_n(t-v))^\beta \le \left(G_n(t+u) - G_n(t)\right) \mathbb{1}_{\{G_n(t)>G_n(t-v)\}}.$$

Next, Lemma 2.5 in Kurtz [9] implies that $G_n(t) \le t$ and that if $G_n(t) > G_n(t-v)$, then $G_n(t) > t - v$: therefore,

$$\mathbb{E}\left[q(Y_n(t+u), Y_n(t))^\beta \mid \mathcal{F}^n_{\gamma_n(t)}\right] q(Y_n(t), Y_n(t-v))^\beta \le v + u,$$

where this inequality holds for any $n \ge 1$ and any $0 \le v \le t$ and $u \ge 0$. These arguments also imply that

$$\mathbb{E}\left[q(Y_n(\delta), Y_n(0))^\beta\right] \le G_n(\delta) \le \delta,$$

and these two inequalities imply the desired tightness of (Y_n) by Theorem 3.8.6 in Ethier and Kurtz [5], since (Y_n) also satisfies the compact containment condition.

Third step

Let us now conclude the proof and show that (X_n) is tight recalling that $X_n = Y_n \circ F_n$. Since (Y_n) is tight, assume without loss of generality (by working along appropriate subsequences and using the Skorohod representation theorem) that $Y_n \to Y$ and $F_n \to F$: if Y were constant (except for maybe one jump) on any interval $[u, v]$ on which F^{-1} is constant, then Lemma 2.3(b) in Kurtz [9] would imply that $X_n \to Y \circ F$ and (X_n) would be tight. Thus, for each interval $[u, v]$ on which F^{-1} is constant, to conclude the proof it is enough to show that Y is constant on $[u, v)$.

Let α denote the constant value taken by F^{-1} on $[u, v]$, and consider a sequence (α_n) such that $\alpha_n \to \alpha$, $F_n(\alpha_n) \to F(\alpha)$ and $F_n(\alpha_n-) \to F(\alpha-)$. Fix u', v' with $[u', v'] \subset (u, v)$. Since F^{-1} is constant on $[u, v]$ and takes the value α, we have $F(\alpha-) \le u < v \le F(\alpha)$, and in particular, $F_n(\alpha_n-) < u' < v' < F_n(\alpha_n)$ for n large enough. For these n, F_n^{-1} is constant on $[u', v']$ and since

$$Y_n(t) = \lim_{s \to t+} X_n(F_n^{-1}(s)-),$$

this implies that Y_n for n large enough is constant on $[u', v']$. The convergence $Y_n \to Y$ in the Skorohod topology then implies that Y is constant on any $[u'', v''] \subset (u', v')$. Since $u' < v'$ were arbitrary in $[u, v]$, and since Y is càdlàg, we obtain by letting $u'' \downarrow u$ and $v'' \uparrow v$ that Y is constant on $[u, v)$ as desired.

3 Scaling limits of Galton–Watson processes in varying environment

A Galton Watson branching process (GW process) is an integer-valued Markov chain $(Z(k), k \ge 0)$ governed by the recursion

$$Z(k+1) = \sum_{i=1}^{Z(k)} \xi_{k,i} \tag{3.1}$$

where the $\xi_{k,i}$'s are i.i.d. random variables having as common distribution the so-called *offspring distribution*. See, for instance, Athreya and Ney [1] for a general introduction, and the introduction in Bansaye and Simatos [2] for more references pertained to the following discussion.

GW processes in random environments, where the sequence of offspring distributions is random, have been introduced by Smith and Wilkinson [12] and have recently been intensively investigated. So far, they have mostly been studied from the viewpoint of their long-time behavior and, as far as we know, their scaling limits have only been studied in the finite variance case. This is in sharp contrast with the case of constant environment, where scaling limits have been exhaustively characterized by Grimvall [6]. These scaling limits are called *Continuous State Branching Processes* and we refer to [4] for more details and complements, see e.g. Proposition 4 therein for the stochastic differential equation satisfied by these limits.

This observation was the starting point of our investigation in [2] of the scaling limits of GW processes in varying environments, where the offspring distribution may change from one generation to the next and have unbounded variance. This corresponds to the quenched approach, where one fixes a realization of the sequence of offspring distributions and studies the behavior of the GW process in this (varying) environment.

In particular, we use Corollary 1.3 above in order to show in [2] that the sequence of GW processes in a varying environment (X_n) considered is tight. It relies on the domination of a characteristic triplet associated to the branching mechanism of X_n. More precisely, here the process may explode in finite time and $[0, \infty]$ is endowed with the metric $d(x, y) = |e^{-x} - e^{-y}|$. The Assumption A1 is automatically satisfied since $[0, \infty]$ endowed with d is compact. To apply Corollary 1.3, we prove in [2] that that for each $t \geq 0$, there exists Δ_t such that for any $s \leq y_0 \leq y \leq t$ with $\mu_n(y_0, y] \leq \Delta_t/2$ and $x_0 \in [0, \infty]$,

$$\mathbb{E}\left[d(x_0, X_n(y))^2 \mid X_n(y_0) = x_0\right] \leq 2\Delta_t \mu_n(y_0, y],$$

where μ_n is a positive finite measure linked to the characteristic triplet of the process X_n. In this case Assumption A2” is satisfied with $\eta = \Delta_t^2$, $F^n = 2\Delta_t\mu_n$ and $F = 2\Delta_t\mu$.

In this context and in a large population approximation, each non-critical offspring distribution (i.e., with mean not equal to one) induces a deterministic jump in the limit: if $Z(k)$ is large, then the law of large numbers gives, in view of (3.1), $Z(k+1) - Z(k) \approx \mathbb{E}(\xi_{k,1} - 1)Z(k)$. If the sequence of offspring distributions stems from the realization of a sequence of i.i.d. offspring distributions that may be, with positive probability, non-critical, then we naturally end up in the limit with a time-inhomogeneous Markov process with accumulations of fixed times of discontinuity. Note that the possible *accumulations* of these discontinuities comes from the fact that, in the usual renormalization schemes, time is sped up.

This phenomenon, illustrated on GW processes, is of course not unique to this class of processes. From a high-level perspective, it suggests that in a varying environment mixing critical and non-critical environments, it is natural to expect in the limit time-inhomogeneous Markov processes with accumulations of fixed times of discontinuity. For instance, the above discussion immediately applies to random walks with time-varying step distributions, a topic covered by Jacod and Shiryaev [7]. It is also a very natural framework in population dynamic and evolution. Indeed when considering scaling limits with time acceleration in a varying environment, fixed times of discontinuity accumulate as soon as instantaneous jumps at fixed times are recurrent in the original time scale. In order to illustrate this point, we consider in Section 4 an application of Theorem 1.1 to study logistic birth and death processes, where the environment provokes catastrophes.

4 Tightness of logistic branching processes with catastrophes

To further motivate our conditions for tightness, we show how to apply the results to the scaling limits of logistic branching processes with catastrophes.

A logistic branching process

Consider the following birth-and-death process:

$$z \in \{0,1,2,\ldots\} \longrightarrow \begin{cases} z-1 & \text{at rate } dz + cz^2, \\ z+1 & \text{at rate } bz, \end{cases} \tag{4.1}$$

for some parameters $b, c, d > 0$: b is the per-individual birth rate, d is the per-individual death rate and $c > 0$ is a logistic term which represents competition between individuals. This process is an example of population-dependent branching processes and is also a special case of logistic branching processes. It plays a very important role in population dynamics, where it is probably the simplest model exhibiting a quasi-stationary regime. Simply put, under a suitable scaling, the population size tends to stabilize for a very long time around the value $z^* = (b-d)/c$ that equalizes the birth and death rates.

Its scaling limits are well-known, namely, if Z_n is the above death-and-birth process with parameters $b = \lambda + n\gamma$, $d = \mu + n\gamma$ and $c = \kappa/n$, then the renormalized process $X_n(t) = Z_n(t)/n$ converges weakly to the logistic Feller diffusion, i.e., the unique solution to the following stochastic differential equation:

$$\mathrm{d}X(t) = (\lambda - \mu - \kappa X(t))\, X(t)\mathrm{d}t + \sqrt{\gamma X(t)}\mathrm{d}B(t),$$

with B a standard Brownian motion. See, for instance, [10].

A logistic branching process with catastrophes

There are many different ways to add “catastrophes” to this logistic branching process. For example, a common way is for the catastrophes to occur at the epochs of an independent Poisson process, and for each individual to toss a coin and die with a certain probability. However, we adopt a slightly different framework, technically more convenient and which fulfills our purpose of illustrating the use of Theorem 1.1 on a non-trivial example. Our framework comes from the equivalent description of the Markov process with transition rates (4.1) via a stochastic differential equation, namely, the unique solution to the stochastic differential equation

$$Z(t) = Z(0) + \int_0^t \int_0^\infty \left(\mathbb{1}_{\{u \le bZ(s-)\}} - \mathbb{1}_{\{bZ(s-) < u \le (b+d+cZ(s-))Z(s-)\}} \right) Q(\mathrm{d}s, \mathrm{d}u),\ t \ge 0,$$

where Q is a Poisson point measure on $[0,\infty)^2$ with intensity $\mathrm{d}s \times \mathrm{d}u$. A simple generalization to this dynamic is given by

$$Z(t) = Z(0) + \int_0^t \int_0^\infty \left(\mathbb{1}_{\{u \le bZ(s-)\}} - \mathbb{1}_{\{bZ(s-) < u \le (b+d+cZ(s-))Z(s-)\}} \right) Q(\mathrm{d}s, \mathrm{d}u) \\ - \int_0^t \int_0^1 (1-\theta) Z(s-) q(\mathrm{d}s, \mathrm{d}\theta) \tag{4.2}$$

where q is a deterministic point measure on $[0,\infty) \times [0,1]$ satisfying $q(\{t\} \times [0,1]) \in \{0,1\}$ for every $t \ge 0$. With the additional integral term $\int_0^t \int_0^1 (1-\theta)\, Z(s-) q(\mathrm{d}s, \mathrm{d}\theta)$, if (t, θ) is an atom of q, then Z undergoes a catastrophe at time t and loses a fraction $\theta \in [0,1]$ of its population. Note that Z given by (4.2) is no longer integer-valued, but this definition will be convenient in order to illustrate the use of Theorem 1.1.

In the literature, catastrophes are usually added at random times, say at the instant of a Poisson process. In this case, q would be a Poisson point measure independent of Q, with intensity $\mathrm{d}s \times \mathbb{P}(F \in \mathrm{d}\theta)$ for some random variable $F \in [0,1]$: the above formulation would then correspond to the *quenched* approach, working conditionally on the random environment. Let us finally mention that this example could be generalized in a number of ways, for instance by considering positive jumps at fixed times of discontinuity or multiple simultaneous births, but here we restrict ourselves to the simplest non-trivial example where we believe that Theorem 1.1 is useful.

We now consider the same scaling as previously, and we write now the birth and death rates for the scaled population:

$$b_n(x) = (\lambda + n\gamma)nx, \quad d_n(x) = \Big(\mu + n\gamma + \frac{\kappa}{n}nx\Big)\, nx. \tag{4.3}$$

For each $n \geq 1$, we also consider a measure q_n with $q_n(\{t\} \times [0,1]) \in \{0,1\}$, and we consider Z_n the solution to (4.2) with these parameters and with initial condition $Z_n(0) = x_0 n$ for some $x_0 \geq 0$. We finally consider the renormalized process

$$X_n(t) = \frac{Z_n(t)}{n},\ t \geq 0,$$

which satisfies the following stochastic differential equation:

$$\begin{aligned} X_n(t) = x_0 &- \int_0^t \int_0^1 \big(1-\theta\big) X_n(s-) q_n(\mathrm{d}s, \mathrm{d}\theta) \\ &+ \int_0^t \int_0^\infty \frac{1}{n} \left(\mathbb{1}_{\{u \leq b_n(X_n(s-))\}} - \mathbb{1}_{\{b_n(X_n(s-)) < u \leq d_n(X_n(s-))\}}\right) Q(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{4.4}$$

Let in the sequel

$$f_n(t) = \int_0^t \int_0^1 (1-\theta) q_n(\mathrm{d}s, \mathrm{d}\theta) \ \text{ and } \ F_n(t) = t + f_n(t).$$

Lemma 4.1. *For $K \geq 0$, let $T_n^K = \inf\{t \geq 0 : X_n(t) \geq K\}$. For any $T \geq 0$,*

$$\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(T_n^K \leq T\right) = 0, \tag{4.5}$$

and for each $K \geq 0$, there exists a constant C_K such that the inequality

$$\mathbb{E}\left[1 \wedge \left(X_n(t \wedge T_n^K) - X_n(s \wedge T_n^K)\right)^2 \mid \mathcal{F}_n(s)\right] \leq C_K \big(F_n(t) - F_n(s)\big) \tag{4.6}$$

holds for all $n \geq 1$ and $0 \leq s \leq t$.

Assuming that $F_n \to F$ (which holds for instance if q_n converges weakly to some measure q), this result gives the tightness of the sequence (X_n), since the assumptions of Corollary 1.2 are then satisfied. Note that F_n and its limit F may be discontinuous, and the upper bound in (4.6) depends on the constant K considered. Also, it is reasonable when $q_n \to q$ to expect any accumulation point to satisfy the following stochastic differential equation

$$\mathrm{d}X(t) = \left(\lambda - \mu - cX(t)\right) X(t)\mathrm{d}t + \sqrt{\gamma X(t)}\mathrm{d}B(t) - \int_0^t \int_0^1 \big(1-\theta\big) X(s-) q(\mathrm{d}s, \mathrm{d}\theta).$$

Proof of Lemma 4.1. The fact that (X_n) satisfies the compact containment condition (4.5) follows from a comparison argument: from (4.3) and (4.4) it follows that

$$X_n(t) \leq X_n(0) + \int_0^t \int_0^\infty \frac{1}{n} \left(\mathbb{1}_{\{u \leq nbX_n(s-)\}} - \mathbb{1}_{\{nbX_n(s-) < u \leq n(b+d)X_n(s-)\}}\right) Q(\mathrm{d}s, \mathrm{d}u)$$

and so classical comparison arguments for stochastic differential equations (see for instance [11, Theorem V.43.1]) imply that $X_n(t) \leq \tilde{X}_n(t)$ with $\tilde{X}_n$ given by

$$\tilde{X}_n(t) = \tilde{X}_n(0) + \int_0^t \int_0^\infty \frac{1}{n} \left(\mathbb{1}_{\{u \leq nb\tilde{X}_n(s-)\}} - \mathbb{1}_{\{nb\tilde{X}_n(s-) < u \leq n(b+d)\tilde{X}_n(s-)\}} \right) Q(\mathrm{d}s, \mathrm{d}u)$$

with $\tilde{X}_n(0) = \lceil nX_n(0) \rceil / n$. One readily checks that $\tilde{X}_n$ is a linear birth and death process (scaled in time and space), whose compact containment condition is easily proved (actually, it is well-known that $(\tilde{X}_n)$ converges weakly to the Feller diffusion). We now turn to the proof of (4.6). The process $((X_n(t), t), t \geq 0)$ is Markov with generator

$$\begin{aligned} \Omega_n(f)(x,t) = \frac{\partial f}{\partial t}(x,t) &+ \left(f\left(x + \frac{1}{n}, t\right) - f(x,t) \right) b_n(x) \\ &+ \left(f\left(x - \frac{1}{n}, t\right) - f(x,t) \right) d_n(x) + \int q_n(\{t\} \times \mathrm{d}\theta) \left(f(\theta x, t) - f(x,t) \right) \end{aligned}$$

and so the stopped process $((X_n(t \wedge T_n^K), t \wedge T_n^K), t \geq 0)$ is Markov and its generator is given by $\Omega_n(f)(x,t)\mathbb{1}_{\{x \leq K\}}$. In particular, for a function f that only depends on x, defining $X_n^K(t) = X_n(t \wedge T_n^K)$,

$$\begin{aligned} \mathbb{E}\left[f(X_n^K(t)) \right] = f(X_n^K(0)) & \\ &+ \int_0^t \mathbb{E}\left[\left(f\left(X_n^K(s) + \frac{1}{n} \right) - f(X_n^K(s)) \right) b_n(X_n^K(s)) \mathbb{1}_{\{T_n^K > s\}} \right] \mathrm{d}s \\ &+ \int_0^t \mathbb{E}\left[\left(f\left(X_n^K(s) - \frac{1}{n} \right) - f(X_n^K(s)) \right) d_n(X_n^K(s)) \mathbb{1}_{\{T_n^K > s\}} \right] \mathrm{d}s \\ &+ \int_0^t \int q_n(\mathrm{d}s \times \mathrm{d}\theta) \mathbb{E}\left[\left(f(\theta X_n^K(s)) - f(X_n^K(s)) \right) \mathbb{1}_{\{T_n^K > s\}} \right]. \end{aligned}$$

For $f(x) = (x - X_n^K(0))^2$, we find after some computation

$$\begin{aligned} \mathbb{E}\left[\left(X_n^K(t) - X_n^K(0) \right)^2 \mid X_n^K(0) \right] &= 2(\lambda - \mu) \int_0^t \mathbb{E}\left[X_n^K(s)(X_n^K(s) - X_n^K(0)) \mathbb{1}_{\{T_n^K > s\}} \right] \mathrm{d}s \\ &+ \frac{\lambda + \mu + 2\gamma n}{n} \int_0^t \mathbb{E}\left(X_n^K(s) \mathbb{1}_{\{T_n^K > s\}} \right) \mathrm{d}s \\ &+ \frac{c}{n} \int_0^t \mathbb{E}\left(X_n^K(s)^2 \mathbb{1}_{\{T_n^K > s\}} \right) \mathrm{d}s \\ &- 2c \int_0^t \mathbb{E}\left[X_n^K(s)^2 (X_n^K(s) - X_n^K(0)) \mathbb{1}_{\{T_n^K > s\}} \right] \mathrm{d}s \\ &- \int_0^t \int q_n(\mathrm{d}s \times \mathrm{d}\theta)(1 - \theta^2) \mathbb{E}\left[X_n^K(s)^2 \mathbb{1}_{\{T_n^K > s\}} \right] \\ &+ \int_0^t \int q_n(\mathrm{d}s \times \mathrm{d}\theta) 2(1 - \theta) \mathbb{E}\left[X_n^K(s) X_n^K(0) \mathbb{1}_{\{T_n^K > s\}} \right]. \end{aligned}$$

Since $f(X_n(t)) = 0$ for $X_n(0) > K$, we can assume that $X_n(0) \leq K$ and we get

$$\begin{aligned} \mathbb{E}\left[\left(X_n^K(t) - X_n^K(0) \right)^2 \mid X_n^K(0) \right] \leq 2|\lambda - \mu| K^2 t + \frac{(\lambda + \mu + 2\gamma n) K t}{n} + \frac{c}{n} K^2 t + 2cK^3 t & \\ + 2K^2 \int_0^t \int (1 - \theta) q_n(\mathrm{d}s \times \mathrm{d}\theta). & \end{aligned}$$

Since all sequences involved are bounded, the result follows. $\square$

References

[1] K. B. Athreya and P. E. Ney, *Branching processes*, Dover Publications Inc., Mineola, NY, 2004, Reprint of the 1972 original [Springer, New York; MR0373040]. MR-2047480

[2] Vincent Bansaye and Florian Simatos, *On the scaling limits of Galton–Watson processes in varying environments*, Electron. J. Probab. **20** (2015), no. 75, 1–36.

[3] Patrick Billingsley, *Convergence of probability measures*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999. MR-1700749

[4] Ma. Emilia Caballero, Amaury Lambert, and Gerónimo Uribe Bravo, *Proof(s) of the Lamperti representation of continuous-state branching processes*, Probab. Surv. **6** (2009), 62–89. MR-2592395

[5] Stewart N. Ethier and Thomas G. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986, Characterization and convergence. MR-838085

[6] Anders Grimvall, *On the convergence of sequences of branching processes*, Ann. Probab. **2** (1974), 1027–1045. MR-0362529

[7] Jean Jacod and Albert N. Shiryaev, *Limit theorems for stochastic processes*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003. MR-1943877

[8] Thomas G. Kurtz, *Semigroups of conditioned shifts and approximation of Markov processes*, Ann. Probab. **3** (1975), no. 4, 618–642. MR-0383544

[9] Thomas G. Kurtz, *Random time changes and convergence in distribution under the Meyer-Zheng conditions*, Ann. Probab. **19** (1991), no. 3, 1010–1034. MR-1112405

[10] Amaury Lambert, *The branching process with logistic growth*, Ann. Appl. Probab. **15** (2005), no. 2, 1506–1535. MR-2134113

[11] L. C. G. Rogers and David Williams, *Diffusions, Markov processes, and martingales. Vol. 2: Itô calculus*, John Wiley & Sons Inc., New York, 1987.

[12] Walter L. Smith and William E. Wilkinson, *On branching processes in random environments*, Ann. Math. Statist. **40** (1969), 814–827. MR-0246380

[13] Ward Whitt, *Some useful functions for functional limit theorems*, Math. Oper. Res. **5** (1980), no. 1, 67–85. MR-561155

[14] Ward Whitt, *Stochastic-process limits*, Springer Series in Operations Research, Springer-Verlag, New York, 2002, An introduction to stochastic-process limits and their application to queues. MR-1876437

Acknowledgments. This work was funded by project MANEGE "Modèles Aléatoires en Écologie, Génétique et Évolution" 09-BLAN-0215 of ANR (French national research agency), Chair Modélisation Mathématique et Biodiversité VEOLIA-Ecole Polytechnique-MNHN-F.X. the professoral chair Jean Marjoulet, and National Science Foundation Grant, DMS 11-06424. While most of it was carried out, F. Simatos was affiliated with Inria.